Calculation of Gear Trains by Approximation[1]
(translated by John G. Kirk)

This paper provides an understanding of the method of calculation of gear trains discovered by Louis-Achille [Brocot]. It has the further advantage of being short and concise, but the reader must have a taste for mathematics to appreciate it. This paper was presented by Achille to the Society of Horologers on 10 June 1860, then to the Society for Encouragement of French Industry on 18 July 1860.[2]

Gentlemen,

I wish to describe a new method for calculating gear trains by approximation. However, before doing so I would like to tell you how and when I was led to this discovery.

Several years ago, I was asked to repair some complicated clocks with astronomical displays. Among these some of the displays had been stopped, and now the owner wished to have them restored to their original state. The means I had available to accomplish this seemed sufficient. I thought that by consulting books on horology I would be able to accomplish this as quickly and accurately as possible.

I had studied Janvier particularly as being the author most concerned with such questions, and tried diligently to understand what he had written. My efforts were completely useless because the methods described in his work were not at all practical, serving only to frustrate those who didn't understand them at all.

After many trials I managed to find the numbers I needed, but, as a first step, I saw quickly that [finding] the best approximation depended above all on skill in arranging the calculations, and I thought then that it would be interesting and useful to create a method free of trial and error, capable of finding convenient and practical results directly.

I considered several procedures and abandoned them in succession as not at all meeting the task I had proposed for myself. Finally, my research was rewarded with a complete success, but once the method was created, I am left with a difficult task to accomplish, the theoretical development of this method. That task is now completed, and ready to be presented.

The presentation is in three parts:

The first is entirely theoretical, where the propositions that are the bases of the operations are presented.

In the second, the operations are indicated in the most practical manner I have been able to devise, followed by numerous examples of their application.

Finally, the third consists of a table for converting decimal fractions into common fractions for the purpose of facilitating the work and to shorten part of the calculations.

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[1] This is the text of a method for calculating useful gear trains to match desired input/output ratios devised by Achille Brocot, as translated from the French by John G. Kirk. [translator's note]

[2] This is the foreword, in the author's italics, to the text of Brocot's paper as provided on pp. 187 – 197 of Richard Chevigny's *Les Brocot, une dynastie d'horlogers,* Editions Antoine Simonin, Neuchâtel [Switzerland] (1991). [translator's note]

The theoretical statement would require quite long development and the purpose of this meeting allows me too little time to address it here; I am thus obliged to limit myself to several practical examples.

It is well-known that when two gears are in mesh, the gear with the smaller number of teeth turns more rapidly than the gear with the larger number. So, taking for example a wheel of 48 [teeth] and a pinion of 6 [leaves], the latter will turn 8 times for each turn of the 48 because 48 divided by 6 gives 8 as the quotient. The ratio of numbers of turns between two arbors can thus be represented as

$$\frac{48}{6} = 8.$$

Also, if the time required for a 6 [leaf] pinion to make one turn is known, it is clear that a wheel of 48 [teeth], which is 8 times that on the pinion, will take 8 times as long to make one turn, so the ratio of times is, again

$$\frac{48}{6} = 8.$$

In the first case, the ratio is the inverse of the tooth count[3]; in the second it is direct[4].

This theorem, which is easily understood in the simple case of a train of two gears, is equally true for any number of gears in mesh. The number of turns of the first and the last arbors are in exactly the same ratio as those of a train of two gears in which the first has the product of the teeth of all of the driving gears and the second the product of the teeth of all of the driven gears.

Consider a train of three arbors composed as follows:

A first wheel of 53 teeth engages a pinion of 8 [leaves], which is attached on a second arbor with a wheel of 41 teeth, with the latter engaging a pinion with 7 [leaves] on a third arbor. In this train, the driving gears are the wheels with 53 and 41 [teeth], and the driven gears the pinions with 8 and 7 [leaves]. Following the principle given above, the ratio of speeds of the wheel with 53 [teeth] and the pinion with 7 [leaves] can be represented as

$$\frac{53 \times 41}{8 \times 7} = \frac{2173}{56} \cong 38.80357.$$

The quotient 38.80357 expresses approximately the number of turns of the pinion of 7 [leaves] for one turn of the wheel of 53 [teeth].

We can see here that it is always easy to determine the ratio of speeds between two arbors when a gear train is described, by taking the product of the numbers of teeth on the driving gears, the product of the numbers of teeth on the driven gears, and dividing the first by the second.

This principle being understood, let us now turn to calculation of the numbers of teeth for gears in a train for which the speeds of the [input and output] arbors are in a given ratio, or very nearly so.

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[3] The ratio is the inverse because the larger number corresponds to the gear that makes the smaller number of turns. [Brocot's note]

[4] The ratio is direct because the large number corresponds to the gear which takes longer to make one turn. [Brocot's note]

We have seen that the ratio given in the form of a common fraction is the exact ratio, while the quotient expressed as a decimal number is often only an approximation. Although the [mathematical] operation is the same in the two cases, it should be done using exact ratios, that is, expressed as common fractions.

First Problem

An arbor turns once in 23 minutes, and we need a suitable train to drive another arbor one turn in 3 hours 11 minutes, or 191 minutes.

Here the ratio of speeds between the two arbors is expressed by the fraction

$$\frac{191}{23}.$$

In examining these numbers, we can see that they are both prime, so to produce the ratio desired we must use a pinion of 23 [leaves] and a wheel of 191 teeth. If the horological application does not allow the use of a wheel with a circumference large enough to have so many teeth, it is clear that we must be satisfied with only an approximation.

The problem resolves to this: Substitute for the ratio

$$\frac{191}{23}$$

another ratio expressed by smaller terms while producing the smallest error possible.

It is clear in either of these expressions that the ratio of number allotted to the pinion [denominator] to the number allotted to the wheel [numerator] should match closely the ratio of 191 to 23.

Dividing 191 by 23, we have a quotient a bit more than 8, so that the ratio desired ratio is between 8:1 and 9:1. If a wheel is given with 8 times as many teeth as the pinion, it will turn once in 8 times longer than the latter, in 8 × 23 = 184 minutes. The resulting error from using the 8:1 ratio is 191 – 184 = 7 minutes[5].

Choosing a wheel with 9 times as many teeth as in the pinion, it will turn once in 9 times as much time, in 9 × 23 = 207 minutes. With this ratio the error is 207 – 191 = 16 minutes.

Note in passing that the division of 191 by 23 gives 8 as the quotient with a remainder 7, so the true quotient is

$$8+\frac{7}{23} \text{ or } 9-\frac{16}{23}.$$

The latter fraction is the complement of the former. The errors in minutes are represented by the numerators 7 and 16.[6]

Let us take the numerator 7 from the 8 : 1 approximation and the numerator 16 from the 9 : 1 approximation and construct the following table, in which the numbers in the first column

[5] The common definition of error is the observed/calculated value minus the true value, so the error here should be "-7 minutes". Brocot follows this definition in his tables below, but doesn't do so consistently in the text. [translator's note]
[6] Considering the previous footnote, these should be "-7" and "16". [translator's note]

apply to the wheels, and in the second to the pinions, and, finally, those in the third used to calculate intermediary approximations:

(1.)

8 : 1	–	7
…………		
…………		
9 : 1[7]	+	16

Subtract 7 from 16 leaving 9, and write this difference[8] above the 16 and put the ratio 17 : 2 in the first two columns. These are the results of adding term by term the ratios 8 : 1 and 9 : 1, so our table becomes:

(2.)

8 : 1	–	7
………….		
17 : 2	+	9
9 : 1	+	16

Operate on the 8 : 1 approximation and the new 17 : 2 approximation as we did on the original two, subtracting 7 from 9 leaving 2, writing this difference above the 9, and putting the approximation 25 : 3, the term by term sums of 8 : 1 and 17 : 2, so our table becomes:

(3.)

8 : 1	–	7
…………..		
25 : 3	+	2
17 : 2	+	9
9 : 1	+	16

This time, the remainder 2 is smaller than 7, so we subtract it first from the 7 and successively the remainders[9], each time adding term by term the approximations to which they correspond as in the following tables.

(4.)

8 : 1	–	7
33 : 4	–	5
……………		
……………		
……………		
25 : 3	+	2
17 : 2	+	9
9 : 1	+	16

(5.)

8 : 1	–	7
33 : 4	–	5
58 : 7	–	3
……………		
……………		
25 : 3	+	2
17 : 2	+	9
9 : 1	+	16

(6.)

8 : 1	–	7
33 : 4	–	5
58 : 7	–	3
83 : 10	–	1
…………...		
25 : 3	+	2
17 : 2	+	9
9 : 1	+	16

These six tables are given only to make clear the process of the calculation, but it can be seen that we will be able to reach our goal directly. By continuing operations on the last table as before, we obtain the following complete table, which is arranged in the order of the approximations to the given ratio.

7 This number is misprinted as "3" in Chevigny. [translator's note]
8 This might be better expressed as "Add -7 and +16 obtaining +9, and write this sum…" [translator's note]
9 See previous footnote. [translator's note]

(7.)

8 :	1	–	7
33 :	4	–	5
58 :	7	–	3
83 :	10	–	1
191 :	23		0
108 :	13	+	1
25 :	3	+	2
17 :	2	+	9
9 :	1	+[10]	16

In these results we recover the given ratio, $\frac{191}{23}$. The approximations given above this fraction have values smaller and the ones below larger than the desired ratio. Let us now consider one of these approximations and calculate its error.

For the approximation 58 : 7, a wheel of 58 [teeth] and a pinion of 7 [leaves], when the latter makes one turn in 23 minutes, we can determine the time taken by the wheel of 58 [teeth] to make its turn by solving the following proportion:

$$7 : 23 :: 58 : \frac{23\times 58}{7}\text{; the last term } = \frac{1334}{7} = 190 + \frac{4}{7}\text{ minutes}$$ [11]

But, as stated above, this wheel should turn once in 191 minutes, so the error is

$$[-]\frac{3}{7}\text{ of a minute}$$ [12].

Taking the approximation 25 : 3, the degree of approximation can be found from the proportion

$$3 : 23 :: 25 : \frac{23\times 25}{3}\text{ ; the last term } = \frac{575}{3} = 191 + \frac{2}{3}\text{ minutes}$$

It can be seen that the error for an approximate ratio is given by the fraction of a minute having the number in column three as the numerator and the number in column one as the denominator, so calculating the error is not necessary[13].

Also, using a pinion of 10 [leaves] and wheel of 83 teeth, the error in the times of rotation will be $[-]\frac{1}{10}$ of a minute[14].

[10] This symbol is misprinted as “–“ in Chevigny. [translator’s note]

[11] The term after the last equals sign is given as “190’’’ + 4/7” in Chevigny. [translator’s note]

[12] As discussed in footnote 5, this fraction should have a minus sign. [translator’s note]

[13] This definition of error is consistent with that in footnote 5 if the sign of the number in column three is included. [translator’s note]

[14] See footnotes 5 and 13. [translator’s note]

Finally, taking a pinion of 13 [leaves] and a wheel of 108 [teeth], the error will be only $\frac{1}{13}$ of a minute, and is in this case the best approximation that can be achieved with a train of two gears.

For a more exact approximation, a train of three arbors is needed with two wheels and two pinions by using a ratio with terms [each] with two convenient factors.

Taking from the list of approximations found above the exact ratio, $\frac{191}{23}$, and each of the two approximations with the least error, $\frac{83}{10}$ and $\frac{108}{13}$, we can calculate intermediate values by following exactly the same process used in creating the previous tables.

We thus have two series of results, those of the first comprising approximations smaller [than the target] and the second approximations larger [than the target], but with such errors always becoming smaller as larger numbers are used.

Here are the two series:

(8.)	83 : 10	– 1	(9.)	108 : 13	+ 1
	274 : 33	– 1	e)	299 : 36	+ 1
a)	465 : 56	– 1		490 : 59	+ 1
	656 : 79	– 1		681 : 82	+ 1
b)	847 : 102	– 1	f)	872 : 105	+ 1
	1038 : 125	– 1		1063 : 128	+ 1
	1229 : 148	– 1		1254 : 151	+ 1
c)	1420 : 171	– 1	g)	1445 : 174	+ 1
	1611 : 194	– 1		1636 : 197	+ 1
d)	1802 : 217	– 1	h)	1827 : 220	+ 1
	1883 : 240	– 1		2018 : 243	+ 1
	…… : …………			…… : …………	
	…… : …………			…… : …………	
	…… : …………			…… : …………	
	…… : …………			…… : …………	
	191 : 23	– 0		191 : 23	+ 0

In these two tables, each of the numbers in the first column represent the product of the [teeth in each of the] wheels, those in the second the product of the [leaves in each of the] pinions. We cannot use these approximations unless the largest prime factor for the numbers in each column cannot be applied respectively to a wheel and to a pinion[15].

Gear trains found from Table 8

a) $\frac{465}{56} = \frac{Wheels}{Pinions}\frac{5\cdot 93}{7\cdot 8}$

b) $\frac{847}{102} = \frac{W.}{P.}\frac{11\cdot 77}{6\cdot 17}$

c) $\frac{1420}{171} = \frac{W.}{P.}\frac{20\cdot 71}{9\cdot 19}$

d) $\frac{1802}{217} = \frac{W.}{P.}\frac{34\cdot 53}{7\cdot 31}$

Gear trains found from Table 9

e) $\frac{299}{36} = \frac{Wheels}{Pinions}\frac{13\cdot 23}{6\cdot 6}$

f) $\frac{872}{105} = \frac{W.}{P.}\frac{8\cdot 109}{3\cdot 35}$

g) $\frac{1445}{174} = \frac{W.}{P.}\frac{17\cdot 85}{6\cdot 29}$

h)[16] $\frac{1827}{220} = \frac{W.}{P.}\frac{29\cdot 63}{11\cdot 20}$

If, among the ratios found none are usable, additional intermediaries can be created by adding two consecutive ratios term by term, but in this case the approximation may not be in proportion to the magnitude of the numbers used, and the resulting error from using these intermediary ratios is represented by a fraction having the sum of the numbers in the first[17] column as numerator and the sum of the numbers in the second column as denominator. [Brocot's italics]

Example [from a) in Table 8]

$$\frac{465+656}{56+79} = \frac{1121}{135} = \frac{W.}{P.}\frac{19\cdot 59}{5\cdot 27} \quad \text{error} \quad \frac{1+1}{56+79} =^{18} [-]\ \frac{2}{135}$$

We have resolved the problem posed of rendering [in a train of gears] the ratio expressed by the common fraction 191/23. We do not quite have the result for the quotient of these two numbers expressed as [the decimal fraction] 8.30435.[19] This problem can be stated as:

A convenient train is needed for which [one turn of] the first arbor turns the second 8.30435 [turns]. The ratio of the number of teeth of the wheel to the number of leaves in the pinion is between 8:1 and 9:1.

The first ratio has an error of –0.30435

The second ratio has an error of +0.69565

[15] These [prime] factors can be found easily by using a table of factors such as that of Burkardt, for example, which is the most accurate and gives the smallest prime factors of numbers up to three million. [Brocot's note]
Burkhardt's table omitted entries with smallest factors of 2, 3, or 5, and was published in Paris in sections in the years 1814 – 1817. Further discussion of Burkhardt's work, as well as yet another variant of the spelling of his name (Burckhardt), can be found at: http://www.1911encyclopedia.org/Mathematical_Table. [translator's note]

[16] The text has "43", not "63", for the last factor in the numerator. [translator's note]

[17] Text has "third" not "first". [translator's note]

[18] Text has "+", not "=". [translator's note]

[19] The phrase "nous aurions pu avoir à le résoudre" is duplicated in the text [translator's note]

The second error is the complement of the first. We proceed again as above, associating the number [–]0.30435 with the ratio 8:1 and the number 0.69565 with the ratio 9:1, as in the following table:

8 : 1	–0.30435
………………………...	
………………………...	
9 : 1	+0.69565

The intermediaries are then calculated by subtracting[20] successively the numbers in the third column and adding successively the numbers term by term in the first and second columns, thus producing a table corresponding to that calculated using common fractions (see Table 7).

The table of conversions from decimal fractions to common fractions remains to be discussed. This table comprises all proper fractions with denominators not greater than 100 arranged in order of increasing value with each calculated to the tenth decimal [place].

Here is how, with the aid of this table, needed numbers can be found rapidly. Consider again the ratio 8.30435. Searching in the table, the fraction [in this ratio] is found to be 7/23. For the ratio 1:8.30435 one can then substitute

$$1 : 8 + \frac{7}{23} \text{ or } 23 : 191,$$

but we have seen that these numbers will not work because 191 is prime and too large [a tooth count] for a wheel.

Taking then the two fractions 7/23 is between, 24/79 and 25/82, we have the following two new ratios

$$1 : 8 + \frac{24}{79} \text{ or } 79 : 656 \text{ and } 1 : 8 + \frac{25}{82} \text{ or } 82 : 681.$$

The first has a negative error, and when combined with 23 : 191 in successive additions, the same results as in Table 8 are recovered. The second has a positive error, and its combination with 23 : 191 will give the same results as in Table 9.

Let us now use the table of fractions to solve the following problem.

Second Problem

An arbor turns once in 24 hours, or one day. A convenient train is required that makes another arbor turn once in 87.96926 days, the [tropical] period of Mercury. Searching in the table, the fraction 0.96926 is not found, but the two consecutive best fractions are

$$87 + \frac{63}{65} : 1, \text{ which gives } 5718 : 65 \text{ and}$$

$$87 + \frac{95}{98} : 1, \text{ which gives } 8621 : 98.$$

[20] “adding algebraically” would be preferable to “subtracting”, see footnote 8. [translator’s note]

Add these two results term by term to obtain the intermediary 14339 : 163, as in the following table

$$\begin{array}{r} 5718 : 65 \\ 14339 : 163 \\ 8621 : 98 \end{array}$$

Adding term by term two consecutive results in the table above gives two new intermediaries and the table becomes

$$\begin{array}{r} 5718 : 65 \\ 20057 : 228 \\ 14339 : 163 \\ 22960 : 261 \\ 8621 : 98 \end{array}$$

In these five results, only one can provide a usable train,

$$\frac{22960}{261} = \frac{Wheels}{Pinions}\frac{112 \cdot 205}{9 \cdot 29} = \frac{W.}{P.}\frac{10 \cdot 56 \cdot 41}{3 \cdot 3 \cdot 29}.$$ [21]

Dividing 22960 by 261 to find the degree of approximation from these numbers, the ratio is 87.96935 with an error of + 0.00009.

I limit myself here to these two examples.

This cursory review of the method I propose seems to me to be sufficient to understand its usefulness, but I think it may need further development to allow the method to be applied to all cases that may arise. Meanwhile, until I deliver my work for printing, I place myself at the disposal of those of my colleagues who need to calculate a gear train, to provide them with the clarifications needed to make the calculation.

Achille Brocot

[21] Actually, the first two terms in the denominator would have to be multiplied by 2 to have a usable leaf count for each pinion. This would have to be compensated by multiplying one term in the numerator by 4 or two terms by 2 each. [translator's note]