VAN DER POL'S EQUATION

Analytic Method of General Solution

By Ziya Akcasu

(Electronics Department, University of Southampton)

SUMMARY.—A new method for solving a quasi-linear equation of the form $x'' - x = \mu f(x)x'$ is proposed. The solution of the first approximation is obtained; it is compared with those given by others[1,2,3,4] and found to be identical with them. The method is applied then to the second approximation and yields the interdependence between the amplitude and instantaneous frequency during the transient period. Finally, the effect of the harmonic content is calculated in the transient state as well as in the steady-state. The results are applied to a thermionic generator.

1. Introduction

In this paper an attempt is made to solve a non-linear equation of the form

$$\frac{d^2x}{dt^2} + x = \mu f(x) \frac{dx}{dt} \quad \ldots \quad \ldots \quad (1)$$

where μ is a real number assumed to be small compared with unity, and where $f(x)$ is an analytical function of x. This equation represents a typical and general class of non-linear differential equations encountered in electrical circuits containing a non-linear dissipative element. In many applications, however, the original equation obtained directly from the circuit is of the integral form of the above, viz.,

$$\frac{dx}{dt} + \int x\, dt = \mu F(x) \quad \ldots \quad \ldots \quad (2)$$

where

$$F(x) = \int f(x) dx \ldots \quad \ldots \quad \ldots \quad (3)$$

$F(x)$ represents the current–voltage characteristic of the non-linear resistive element. As will be seen from the application given at the end of this paper, no constant term appears in this original equation. But generally this integro-differential equation is not suitable because of the integral term. Therefore, it is usually differentiated with respect to t. This leads to the equation (1) which is a special type of the more general non-linear differential equation

$$\frac{d^2x}{dt^2} + x = \mu f\left(x, \frac{dx}{dt}\right) \ldots \quad \ldots \quad (4)$$

For this equation there exist several quantitative methods yielding approximate solutions.

In these methods it is generally assumed that the solution of the equation is of the form

$$x = a \sin (t + \theta) \quad \ldots \quad \ldots \quad (5a)$$

where a and θ are unknown functions of t. (For the sake of simplicity of expressions one sets $\omega = 1$, hence works with the normalized frequency; see later.) It is obvious that this is equivalent to

$$x = b_1 \sin t + b_2 \cos t \quad \ldots \quad \ldots \quad (5b)$$

where b_1 and b_2 are again functions of t. Either of these solutions is substituted into the equation (1) and by this means two auxiliary equations are obtained from which a, θ or b_1, b_2 can be found.

In this paper the same procedure will be followed, but the solution of 5(a) or 5(b) will be substituted into the equation (2) instead of into equation (1).

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2. First Approximation

If the solution of the form 5(a) is substituted into (2) the first term becomes

$$\frac{dx}{dt} = \frac{da}{dt} \sin (t + \theta) + a\left(1 + \frac{d\theta}{dt}\right) \cos (t + \theta) \quad (6)$$

For the second term the integral $\int a \sin (t + \theta)dt$ must be evaluated. For this case, however, it is more convenient to use the solution of the form 5(b); e.g.,

$$\int(b_1 \sin t + b_2 \cos t)dt = - b_1 \cos t + b'_1 \sin t - \int b''_1 \sin t\, dt + b_2 \sin t + b'_2 \cos t - \int b''_2 \cos t\, dt \quad \ldots \quad \ldots \quad \ldots \quad (7a)$$

or

$$\int [(b_1 + b''_1) \sin t + (b_2 + b''_2) \cos t]dt = (- b_1 + b'_2) \cos t + (b_2 + b'_1) \sin t \quad (7b)$$

where the dashes show the derivatives with respect to t. So far no assumption has been made about b_1 and b_2. Now it is assumed that b_1 and b_2 vary slowly with time. As will be seen later, the second derivatives of b_1 and b_2 are of the order of μ^2. As a first approximation b''_1 and b''_2 will be neglected in comparison with b_1 and b_2.

Substituting $b_1 = a \cos \theta$, $b_2 = a \sin \theta$ into (7) there results

$$\int a \sin (t + \theta)dt = \left(- a + a\frac{d\theta}{dt}\right) \cos (t + \theta) + \frac{da}{dt} \sin (t + \theta) \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

The third term gives:

$$F[a \sin (t+\theta)] = \phi_1(a) \sin (t+\theta) + \phi_2(a) \sin [2(t+\theta)] + \ldots + \psi_0(a) + \psi_1(a) \cos (t+\theta) + \psi_2(a) \cos [2(t+\theta)] + \quad (9)$$

where ϕ_i, ψ_i are Fourier coefficients of the function $F(a \sin u)$.

By combining the equations (2), (6), (8) and (9) and equating the coefficients of $\sin (t+\theta)$ and $\cos (t+\theta)$ one obtains

$$\frac{da}{dt} = \frac{\mu}{2}\phi_1(a) \quad \ldots \quad (10)$$

$$\frac{d\theta}{dt} = \frac{\mu}{2}\psi_1(a) \quad \ldots \quad (11)$$

where

$$\phi_1(a) = \frac{1}{2\pi}\int_0^{2\pi} F(a \sin u) \sin u \, du \quad \ldots \quad (12a)$$

$$\psi_1(a) = \frac{1}{2\pi}\int_0^{2\pi} F(a \sin u) \cos u \, du = 0 \quad (12b)$$

Here all the harmonic terms have been disregarded.

The equations (10) and (11) are the auxiliary equations from which a and θ are to be solved. From (12b) it follows that $d\theta/dt = 0$; this means that the instantaneous frequency does not change with time.

The equations (10) and (11) are identical with those obtained in a different way by Kryloff and Bogoliuboff.[2] Nothing will be said in this paper about the stability of this solution since this question is widely investigated in the book just mentioned and also in that by Minorsky.[3] But it must be recalled that the equations are obtained by ignoring second derivatives in (7b) and the harmonics in (9). The constant term in (9) represents the well-known detection current due even power terms in $F(x)$. This is also ignored since only the oscillatory solution is considered in this treatment.

3. Second Approximation

To illustrate the effect of the second derivatives of the amplitude, the normal linear differential equation will be first discussed. If it is assumed, that in the equations (2) and (9) $F(x) \equiv x$, then from equations (10) and (11) there follows:

$$\frac{da}{dt} = \frac{\mu}{2}a; \quad \frac{d\theta}{dt} = 0 \quad \ldots \quad (13)$$

If a and θ are solved from these equations and substituted in (5a) one obtains

$$x = C\, e^{\mu t/2} \sin (t + \theta_0) \quad \ldots \quad (14)$$

where C and θ_0 are the constants of integration. On the other hand the exact solution of the linear differential equation

$$x'' + x = \mu x'$$

is known to be

$$x = Ce^{\mu t/2} \sin \left[\left(\sqrt{1 - \frac{\mu^2}{4}}\right)t + \theta_0\right] \quad (15)$$

The two solutions (15) and (16) differ in the frequency. This difference cannot be due to the neglect of harmonics, because there are no harmonics in a linear circuit. Hence the discrepancy can only be due to the neglect of the second derivative in equation (7b). To prove this point a solution of the form $x = a \sin (\omega t + \theta_0)$ is substituted in the above linear equation. In this solution ω and θ_0 are constants and a is a function of time which can easily be determined.

The result is

$$\frac{da}{dt} = \frac{\mu}{2}a$$

$$\omega = \sqrt{1 - \frac{1}{a}\frac{d^2a}{dt^2}} = \sqrt{1 - \frac{\mu^2}{4}} \quad \ldots \quad (16)$$

From this it follows that the frequency of linear oscillations depends on the second derivative of the amplitude. It also follows that the solution of the first approximation (14) is accurate as far as the amplitude is concerned.

If the second derivatives are to be taken into account, integration by parts must be carried one stage further than is done in (7a). Hence

$$\int (b_1 \sin t + b_2 \cos t)\, dt = -b_1 \cos t + b'_1 \sin t + b''_1 \cos t - \int b'''_1 \cos t\, dt + b_2 \sin t + b'_2 \cos t - b''_2 \sin t + \int b'''_2 \sin t\, dt \quad (17)$$

$$\int [(b_1 - b'''_2) \sin t + (b_2 + b'''_1) \cos t]\, dt = (-b_1 + b''_1 + b'_2) \cos t + (b'_1 + b_2 - b''_2) \sin t \quad \ldots \quad (18)$$

It is assumed that

$$\frac{b'''_2}{b_1} \ll 1 \quad \text{and} \quad \frac{b'''_1}{b_2} \ll 1$$

This means that the third derivatives can be neglected in comparison with the original functions. Substituting in (18) $b_1 = a \cos \theta$ and $b_2 = a \sin \theta$

$$\int a \sin (t+\theta) dt = (-a + a\theta' + a'' - a\theta'^2) \cos (t+\theta) + (a' - 2a'\theta' - a\theta'') \sin (t+\theta) \quad \ldots \quad (19)$$

The evaluation of this integral is given in Appendix 1.

By combining the equations (2), (6), (9) and (19), and equating the coefficients of sin and cos terms one obtains

$$\frac{da}{dt} - \frac{1}{2}\frac{1}{a}\frac{d}{dt}\left(a^2\frac{d\theta}{dt}\right) = \frac{\mu}{2}\phi_1(a) \quad \ldots \quad (20)$$

$$\left(\frac{d\theta}{dt}\right)^2 - 2\frac{d\theta}{dt} = \frac{1}{a}\frac{d^2a}{dt^2} \quad \ldots \quad \ldots \quad (21)$$

Since the first approximation has been found sufficiently accurate for obtaining the amplitude, the second term in (20) will be ignored. The first term in (21) will also be neglected since $(d\theta/dt)^2$ is of the order of μ^4.

Instead of using equations (10) and (11) a and θ are solved from the equations

$$\frac{da}{dt} = \frac{\mu}{2}\phi_1(a) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

$$\frac{d\theta}{dt} = -\frac{1}{2}\frac{1}{a}\frac{d^2a}{dt^2} \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

and the values obtained substituted into $x = a \sin(t + \theta)$. Thus an improved solution is found. The instantaneous frequency is obtained by differentiating $(t + \theta)$ with respect to t whence

$$\omega = 1 - \frac{1}{2}\frac{1}{a}\frac{d^2a}{dt^2} \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

It must be realized that ω is the normalized frequency; i.e., $\omega_0^2 = 1/LC$ is supposed to be unity. By substituting d^2a/dt^2 from (22), one obtains

$$\omega = 1 - \frac{\mu^2}{8}\theta_1(a)\frac{d\phi_1(a)}{dt} \quad \ldots \quad \ldots \quad (25)$$

4. Effect of Harmonics

It is seen from the equation (25) that the influence of the amplitude variations on the frequency is of the order of μ^2. On the other hand, it has been shown by Kryloff and Bogoliuboff[2] that the reduction in the frequency due to the harmonics is also of the order of μ^2. It follows that to ignore the harmonics as done in the previous section is not justified if one is interested in frequency changes of the order of μ^2. It is the purpose of this section to take into account the harmonics as well as the second derivatives.

For the sake of simplicity it will be assumed that $F(x)$ is an odd function of x; i.e., that $F(x) = -F(-x)$. In this case the Fourier series of $F(a \sin u)$ contains, as can easily be seen, only odd sine terms. Hence

$$F(a \sin u) = \phi_1(a) \sin u + \phi_3(a) \sin 3u + \phi_5(a) \sin 5a + \ldots \quad \ldots \quad \ldots \quad (26)$$

It is also assumed that in the solution only the third harmonic is pronounced. Then the solution will be of the form

$$x = a \sin(t + \theta) + b \cos[3(t + \alpha)] \quad \ldots \quad (27)$$

where a, θ, b and α are functions of time. It must be borne in mind that $d\alpha/dt$ may not be equal to $d\theta/dt$ during the transient period whereas it is known to be so for the stationary oscillations.

As done previously, the expression (27) for x must be substituted in equation (2).
Then the first term is

$$\frac{dx}{dt} = \frac{da}{dt}\sin(t + \theta) + a\left(1 + \frac{d\theta}{dt}\right)\cos(t + \theta) + \frac{db}{dt}\cos[3(t + \alpha)] - 3b\left(1 + \frac{d\alpha}{dt}\right)\sin[3(t + \alpha)] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

With the help of the formula given in Appendix 1 the second term becomes

$$\int x dt = (-a + a\theta' + a'')\cos(t + \theta) + a\sin(t + \theta) + \frac{b}{3}(1 - \alpha')\sin[3(t + \alpha)] + \frac{b'}{9}\cos[3(t + \alpha)] \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

When evaluating this integral the second derivatives of the third harmonic are neglected because these quantities which are small in themselves appear with a factor 1/9.

The third term in equation (2) requires a comparatively longer calculation. At this stage it is assumed that the amplitude of the third harmonic is small in comparison with that of the fundamental. It follows from this that

$$F[a \sin(t + \theta) + b \cos 3(t + \alpha)] \simeq F[a \sin(t + \theta)] + \frac{dF[a \sin(t + \theta)]}{d[a \sin(t + \theta)]} b \cos 3[(t + \alpha)] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

If the function $\frac{dF(a \sin u)}{d(a \sin u)}$ is expanded into a Fourier series

$$\frac{dF(a \sin u)}{d(a \sin u)} = \psi_0(a) + \psi_2(a)\cos 2u + \psi_4(a)\cos 4u + \cdots \quad \ldots \quad (31)$$

The coefficients can be calculated in terms of those that are already determined in equation (26). If the equation (26) is differentiated with respect to u one obtains

$$\frac{dF(a \sin u)}{du} = \frac{d[F(a \sin u)]}{d(a \sin) u} a \cos u = \phi_1(a)\cos u + 3\phi_3(a)\cos 3u + 5\phi_5(a)\cos 5u + \cdots$$

Dividing this by $(a \cos u)$, evaluating the terms

$$\frac{\cos 3u}{\cos u}, \quad \frac{\cos 5u}{\cos u}, \ldots,$$

and combining it with (31) gives (see Appendix 2)

$$\left.\begin{aligned}
\psi_0(a) &= \frac{1}{a}[\phi_1(a) - 3\phi_3(a) + 5\phi_5(a) \ldots] \\
\psi_2(a) &= \frac{2}{a}[3\phi_3(a) - 5\phi_5(a) + \ldots] \\
\psi_4(a) &= \frac{2}{a}[5\phi_5(a) - 7\phi_3(a) + \ldots]
\end{aligned}\right\} \quad (32)$$

Combining the equations (27), (31) and (2) and denoting $3(\theta - \alpha)$ by δ gives

$$F(x) = \left[\phi_1(a) + \frac{b}{2}[\psi_2(a) - \psi_4(a)] \sin \delta\right] \sin (t + \theta)$$
$$+ \left[3 \frac{b}{a} \phi_3(a) \cos \delta\right] \cos (t + \theta) + \left[b\psi_0(a)\right.$$
$$\left. + \frac{b}{2}\psi_6(a) \cos 2\delta + \phi_3(a) \sin \delta\right] \cos 3 (t + \alpha)$$
$$+ \left[\phi_3(a) \cos \delta - \frac{b\psi_6(a)}{2} \sin 2\delta\right] \sin 3(t + \alpha) \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

Finally, the equations (28), (29) and (33) are substituted into (2) and sin and cos terms are equated. Then

$$\frac{d\theta}{dt} = -\frac{1}{2a}\frac{d^2a}{dt^2} + \frac{3}{2}\mu\frac{\phi_3(a)}{a}\cos \delta \quad \ldots \quad \ldots \quad (34)$$

$$\frac{da}{dt} = \frac{\mu}{2}\left[\phi_1(a) + \frac{b}{a}[\psi_2(a) - \psi_4(a)] \sin \delta\right] \quad \ldots \quad (35)$$

$$b\left(\frac{4}{5} + \frac{d\alpha}{dt}\right) = -\frac{3}{10}\mu\, \phi_3(a) \cos \delta + \frac{\mu}{2} b\, \psi_6(a) \sin 2\delta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

$$\frac{db}{dt} = \frac{9}{10}\mu\left[b\psi_0(a) + \phi_3(a) \sin \delta - \frac{b}{2}\psi_6(a) \cos 2\delta\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

From these equations a, θ, b and α can be found. In equation (35) the last term shows the effect of the third harmonic on the amplitude of fundamental frequency. As will be seen from the formulae (39) and (42), obtained below for b and δ, the magnitude of the second term is of the order of μ^2 and, therefore, can be neglected.

The same applies to $d\alpha/dt$ and to the last term in (36). Hence equation (35) gives

$$\frac{da}{dt} = \frac{\mu}{2}\phi_1(a) \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

from which a can be found as a function of time, and (36) gives b in terms of a; i.e.,

$$b = -\tfrac{3}{8}\mu\, \phi_3(a) \quad \ldots \quad \ldots \quad (39)$$

Here cos δ is set equal to unity. The error due to this approximation is of the order of μ^2 because δ is proportional to μ.

The change in the frequency of the fundamental is obtained by combining the equations (34) and (39), whence

$$\frac{d\theta}{dt} = -\frac{1}{2a}\frac{d^2a}{dt^2} - 4\frac{b^2}{a^2} \quad \ldots \quad (40)$$

The first term on the right-hand side of (40) is the change due to the varying amplitude and therefore applies only to the transient period. The second term shows the effect of the third harmonic. The change in the frequency of the third harmonic (i.e., $d\alpha/dt$) will now be found.

It seems appropriate to evaluate $d\delta/dt$ rather than $d\alpha/dt$, because the former gives, during the transient period, the deviation of the 'third harmonic' from three times the fundamental frequency. For the stationary oscillations it is known that this deviation is zero. It must be recalled that δ was equal to $3(\theta - \alpha)$ and consequently

$$\frac{d\delta}{dt} = 3\left(\frac{d\theta}{dt} - \frac{d\alpha}{dt}\right) \quad \ldots \quad \ldots \quad (41a)$$

or

$$\frac{d\delta}{dt} = 3\omega_f - \omega_h \quad \ldots \quad \ldots \quad (41b)$$

where the subscripts f and h refer to fundamental and harmonic respectively.

By differentiating equation (39) with respect to t and substituting da/dt from (38) one obtains

$$\frac{db}{dt} = -\frac{3}{16}\mu^2\frac{d\phi_3(a)}{da}\phi_1(a)$$

Combining this with (37) gives

$$\delta = \tfrac{3}{8}\mu[\phi_1(a) - 3\phi_3(a) + 5\phi_5(a)] - \frac{5}{24}\mu\frac{\phi_1(a)}{\phi_3(a)}\frac{d\phi_3(a)}{da} \quad \ldots \quad \ldots \quad (42)$$

With the help of the equations (38), (39), (40) and (42) one can determine the unknown quantities in (27) and thus obtain the solution required.

5. Application

Consider a valve oscillator with a tuned anode circuit (Fig. 1). The currents in the circuit are expressed in terms of the p.d. v across C as follows:—

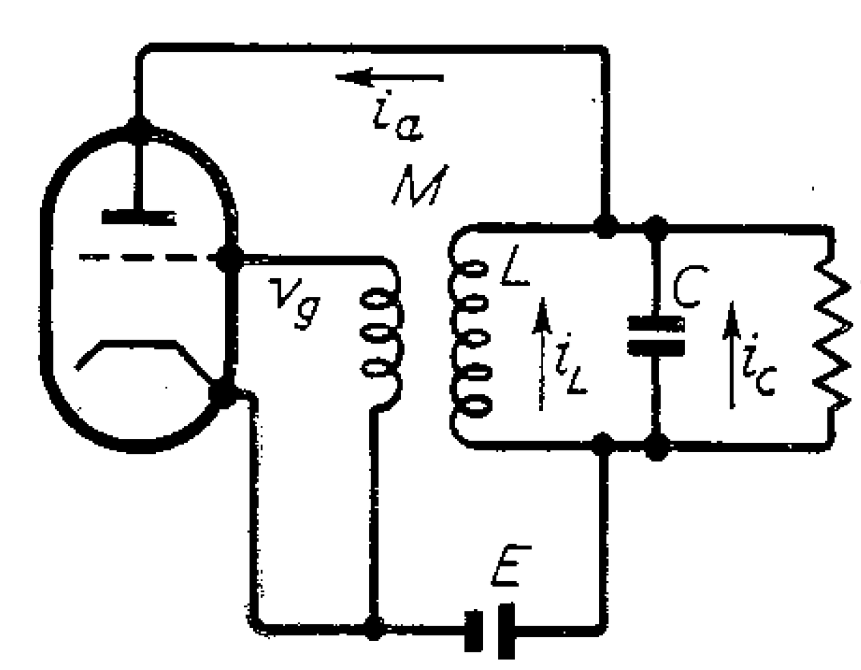

Fig. 1. *Valve oscillator with tuned-anode circuit.*

$$i_R = \frac{v}{R};\; i_c = C\frac{dv}{dt};\; I_L = I_0 + i_L$$
$$= I_0 + \frac{1}{L}\int v\,.\,dt \quad I_a = \psi(v_a + \mu v_g) \quad (43)$$

where I_0 is the d.c. component of the current in the inductance and where the function $\psi(v_a + \mu v_g)$ shows the non-linear valve characteristic. v_a stands for the effective anode voltage; i.e., $v_a = E - v$.

Remembering that $v_g = \frac{M}{L}v$ one can write I_a as follows

$$I_a = \psi(E + kv)$$

where $k = \left(\mu \frac{M}{L} - 1\right)$

Kirchhoff's equation for the currents gives

$$i_C + i_R + I_L = I_a$$

or

$$i_C + i_R + i_L = I_a - I_0 = i_a \qquad (44)$$

where i_a and i_L are the varying components of the currents in the valve and in the inductor respectively.

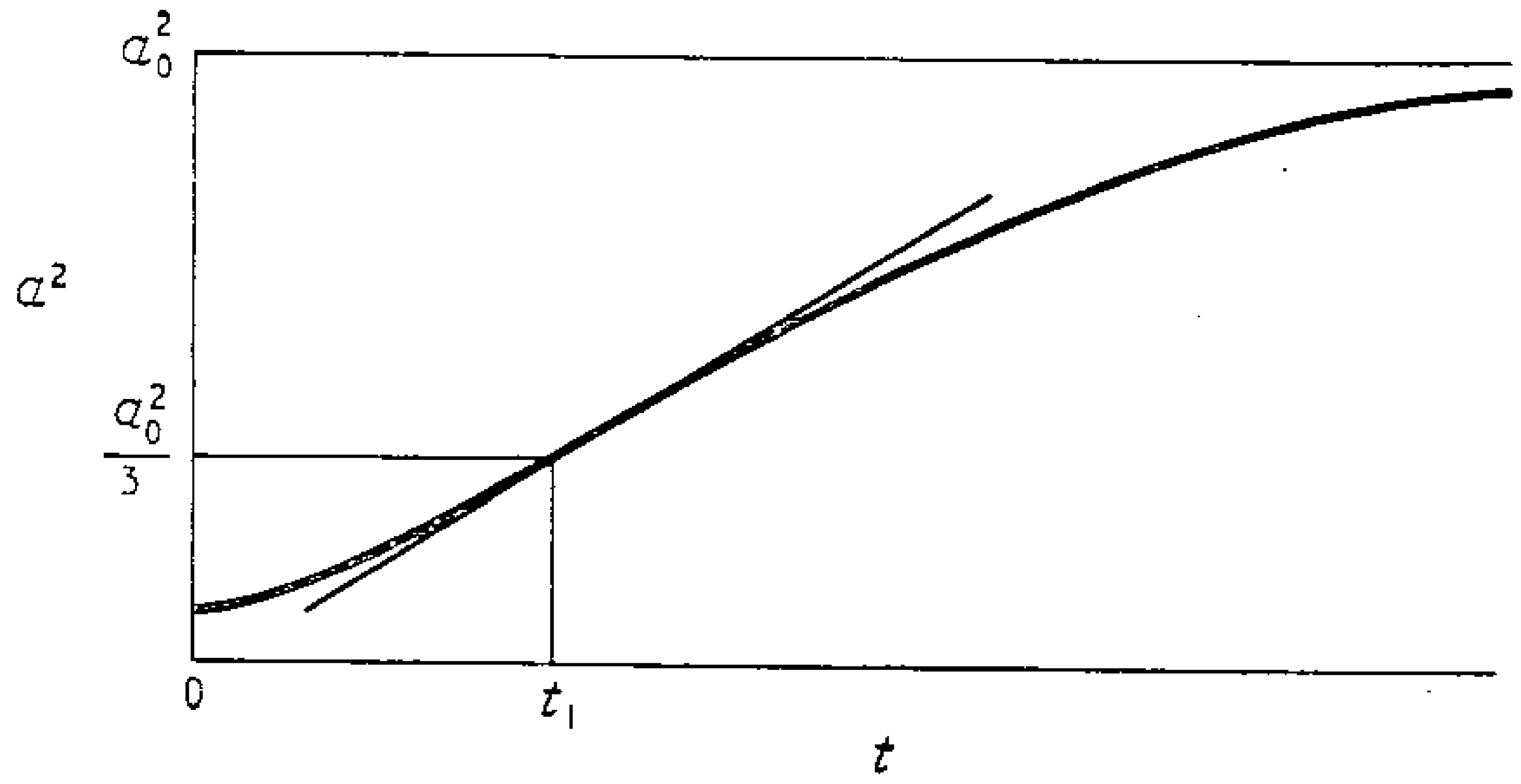

Fig. 2. *This curve represents equation* (51) *for the triode oscillator under consideration.*

Since E is a constant voltage, i_a can be expressed as a function of v; i.e.,

$$i_a = \phi(v) \quad \ldots \quad \ldots \quad \ldots \quad (45)$$

By substituting from (43) into (44) and denoting

$\frac{1}{\omega_0 C}\left[\phi(v) - \frac{v}{R}\right]$ by $\mu F(v)$ one obtains

$$\frac{dv}{dt} + \int v dt = \mu F(v) \quad \ldots \quad \ldots \quad \ldots \quad (46)$$

where t is the normalized time; i.e., $t = \omega_0 \tau$, and $\omega_0^2 = 1/LC$.

It is assumed now that the characteristic of the non-linear resistive element takes the form

$$F(v) = v(1 - \gamma v^2) \quad \ldots \quad \ldots \quad (47)$$

For $v = a \sin u$ (47) becomes

$$F(a \sin u) = a(1 - \tfrac{3}{4}\gamma a^2) \sin u + \frac{\gamma a^3}{4} \sin 3u \quad (48)$$

Thus, the coefficients in equation (26) are now

$$\left.\begin{aligned} \phi_1(a) &= a\left(1 - \frac{a^2}{a_0^2}\right) \\ \phi_3(a) &= \frac{1}{3}\frac{a^3}{a_0^2} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (49)$$

where $a_0^2 = 4/3\gamma$.

From (38) it follows that

$$\frac{da}{dt} = \frac{\mu}{2} a\left(1 - \frac{a^2}{a_0^2}\right) \quad \ldots \quad \ldots \quad (50)$$

From (50)

$$a = \frac{a_0}{\sqrt{1 + c a_0^2 e^{-\mu t}}} \quad \ldots \quad \ldots \quad (51)$$

where c is the constant of integration. For large values of t, a approaches a_0, the amplitude of the steady-state oscillation.

For the amplitude of the third harmonic, equation (39) gives

$$b = -\frac{\mu}{8}\frac{a^3}{a_0^2} \quad \ldots \quad \ldots \quad \ldots \quad (52)$$

The instantaneous angular frequency of the fundamental is obtained from (40) as follows

$$\omega = 1 + \frac{d\theta}{dt} = 1 - \frac{1}{2a}\frac{d^2a}{dt^2} - \frac{\mu^2}{16}\frac{a^4}{a_0^4} \quad \ldots \quad (53)$$

or, evaluating d^2a/dt^2 from (50),

$$\omega = 1 - \frac{\mu^2}{8}\left(1 - \frac{a^2}{a_0^2}\right)\left(1 - 3\frac{a^2}{a_0^2}\right) - \frac{\mu^2}{16}\frac{a^4}{a_0^4} \quad \ldots \quad \ldots \quad \ldots \quad (54)$$

It must be recalled that ω shows the normalized frequency. The curves representing equations (51) and (54) are shown in Figs. 2 and 3 respectively. From the graph it is seen that the frequency of oscillation is at first smaller and then larger than the steady-state value.

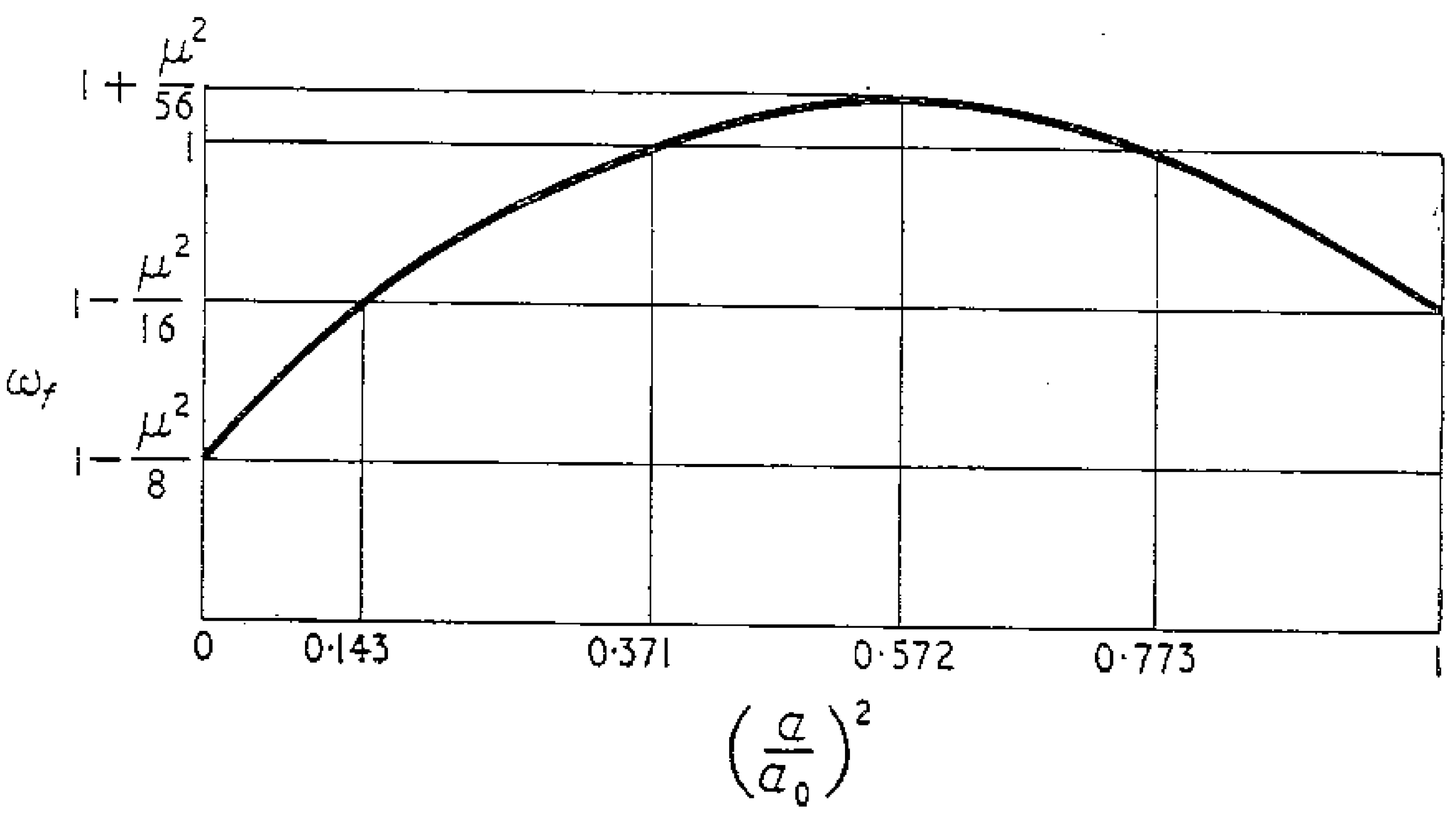

Fig. 3. *Equation* (54) *produces this curve for the example considered.*

To find the frequency of the '3rd harmonic', equation (42) must be investigated. By substituting the terms $\phi_1(a)$ and $\phi_3(a)$ from (49) into (42) one obtains

$$\delta = -\frac{\mu}{8}\left(2 + \frac{a^2}{a_0^2}\right) \quad \ldots \quad \ldots \quad (55)$$

Differentiating (55) with respect to t,

$$\frac{d\delta}{dt} = 3\omega_f - \omega_h, \text{ viz.,}$$

$$\frac{d\delta}{dt} = -\frac{\mu^2}{8}\frac{a^2}{a_0^2}\left(1 - \frac{a^2}{a_0^2}\right) \quad \ldots \quad (56)$$

Fig. 4 shows $-\frac{d\delta}{dt}$ as a function of $\left(\frac{a^2}{a_0^2}\right)$. It follows that the frequency of the '3rd harmonic' is slightly larger than three times that of the fundamental during transient time.

Fig. 5 represents the amplitude of the '3rd harmonic' as a function of $(a/a_0)^2$ during transient time.

The solution for the steady-state oscillation is obtained by replacing a by a_0 in equations (52), (54) and (55).

Hence

$$x = a_0 \sin\left[\left(1 - \frac{\mu^2}{16}\right)t + \theta_0\right] - \frac{\mu a_0}{8}\cos 3\left[\left(1 - \frac{\mu^2}{16}\right)t + \theta_0 + \frac{\mu}{8}\right] \quad (57)$$

With the exception of the last term $\mu/8$ this solution is identical with that obtained by Kryloff and Bogoliuboff.

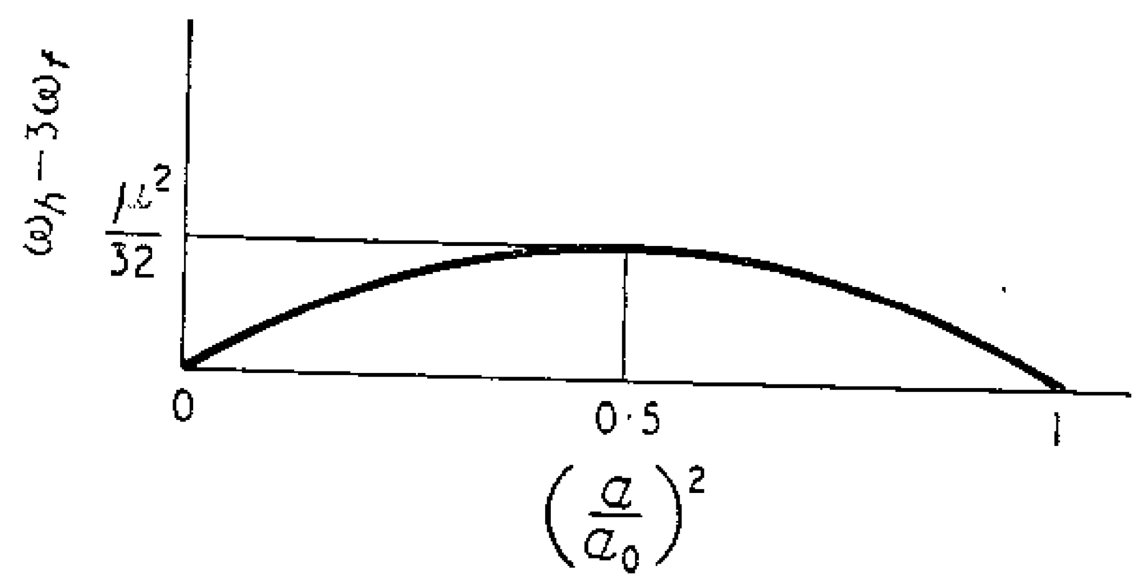

Fig. 4

Fig. 5

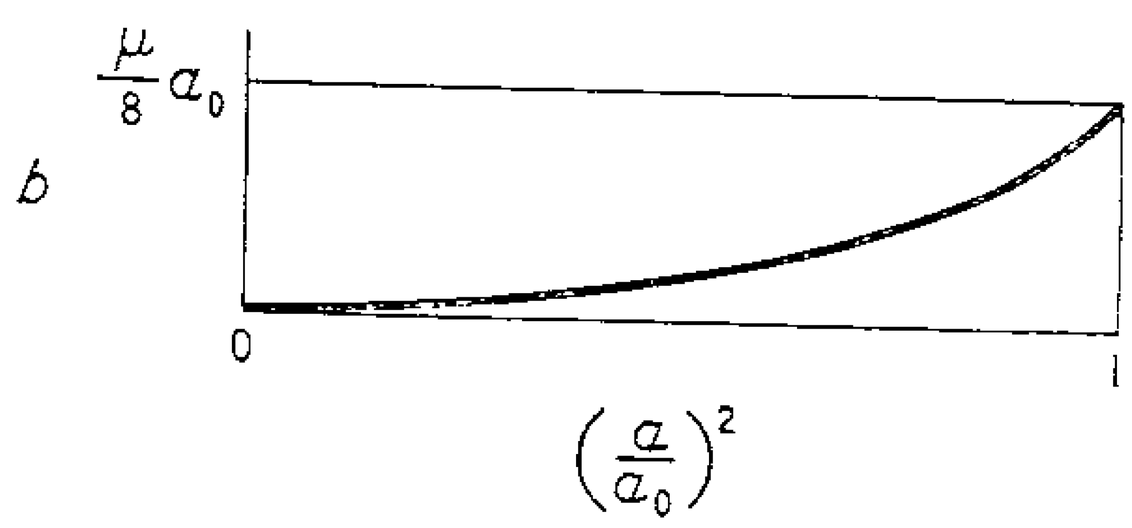

Fig. 4 (above). *Relation between $d\delta/dt$ and a^2/a_0^2.*

Fig. 5 (below). *Amplitude of 'third harmonic'.*

6. Conclusion

The use of the integro-differential equation has made it possible to deal more rigorously than before with conditions during transient period. The resultant formulae apply to the circuits containing the non-linear resistive element, whose current-voltage characteristic is an odd function. Amplitude and frequency of the fundamental and '3rd harmonic' are obtained both for the transient period and for the steady-state. The method can be applied to a resistive characteristic of a more general nature.

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APPENDIX 1

To evaluate the integral $\int a \sin(t + \theta)dt$ it is convenient to define a complex function $X = ae^{j\theta}$

Hence

$$\int a \sin(t + \theta)\, dt = \text{Imag.}\left[\int X e^{jt}\, dt\right]$$

Integrating the latter by parts

$$\int X e^{jt}\, dt = (-jX + X' + jX'')e^{jt} + j\int X''' e^{jt}\, dt.$$

Neglecting jX''' in comparison with X

$$\int X e^{jt}\, dt = (-jX + X' + jX'')e^{jt}.$$

Substituting $X = ae^{j\theta}$ in the last equation

$$\begin{aligned}\int X e^{jt}\, dt &= [-ja + a' + ja\theta' + j(a'' + j2a'\theta' + ja\theta'' - a\theta'^2)]e^{j(t+\theta)}\\ &= j[-a + a\theta' + a'' - a\theta'^2]e^{j(t+\theta)} + [a' - 2a'\theta' - a\theta'']e^{j(t+\theta)}\end{aligned}$$

Retaining only imaginary terms gives

$$\int a \sin(t + \theta)\, dt = [-a + a\theta' + a'' - a\theta'^2]\cos(t + \theta) + [a' - 2a'\theta' - a\theta'']\sin(t + \theta)$$

APPENDIX 2

Let the function $F(a \sin u)$ be an odd function of $a \sin u$. Then

$$F(a \sin u) = \phi_1(a)\sin u + \phi_3(a)\sin 3u + \phi_5(a)\sin 5u + \ldots$$

Differentiation with respect to u gives

$$\frac{dF(a \sin u)}{du} = \frac{dF(a \sin u)}{d(a \sin u)}\, a\cos u = \phi_1(a)\cos u + 3\phi_3(a)\cos 3u + 5\phi_5(a)\cos 5u + \ldots$$

Dividing by $(a \cos u)$ gives

$$\frac{dF(a \sin u)}{d(a \sin u)} = \frac{\phi_1(a)}{a} + 3\frac{\phi_3(a)}{a}\frac{\cos 3u}{\cos u} + 5\frac{\phi_5(a)}{a}\frac{\cos 5u}{\cos u} + \ldots$$

On the other hand

$$\begin{aligned}\cos(2k + 1)u &= \cos u \cos 2ku - \sin u \sin 2ku\\ &= \cos u \cos 2ku - \tfrac{1}{2}[\cos(2k - 1)u - \cos(2k + 1)u]\end{aligned}$$

Hence

$$\frac{\cos(2k + 1)u}{\cos u} = 2\cos 2ku - \frac{\cos(2k - 1)u}{\cos u}$$

This recurrence formula gives for $k = 0, 1, 2 \ldots$

$$1 = 1$$

$$\frac{\cos 3u}{\cos u} = 2\cos 2u - 1$$

$$\frac{\cos 5u}{\cos u} = 2\cos 4u - 2\cos 2u + 1$$

$$\frac{\cos 7u}{\cos u} = 2\cos 6u - 2\cos 4u + 2\cos 2u - 1$$

Substituting these expressions in (1) one obtains

$$\begin{aligned}\frac{dF(a \sin u)}{d(a \sin u)} &= \frac{1}{a}[\phi_1(a) - 3\phi_3(a) + 5\phi_5(a) - \ldots]\\ &- \frac{2}{a}[3\phi_3(a) - 5\phi_5(a) + 7\phi_7(a) - \ldots]\cos 2u\\ &- \frac{2}{a}[5\phi_5(a) - 7\phi_7(a) + 9\phi_9(a)\ldots]\cos 4u + \ldots\end{aligned}$$

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